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Arithmetic Progressions Consisting Only of Primes

By Emil Grosswald and Peter Hagis, Jr.

Abstract. Let $N_m(x)$ denote the number of arithmetic progressions consisting of m primes with largest member not exceeding x. $N_m(x)$ has been tabulated for $3 \leqslant m \leqslant 10$ and selected values of x between 1000 and 50000, and the results are compared here with those obtained by (heuristic) asymptotic approximations to $N_m(x)$.

1. Introduction. Although an old conjecture asserts the existence of arithmetic progressions of arbitrary length and consisting only of primes, the longest known progression of primes consists of only 17 terms and was discovered only recently [11]. In what follows, when we mention an arithmetic progression, or even simply a progression, we shall mean an arithmetic progression all of whose terms are primes. Computer searches for long arithmetic progressions have been made by Golubev ([1], [2], [3], [4]), Karst and Root ([9], [10]), Weintraub ([11], [12]) and others ([7], [8]). While a fairly large number of progressions with 10 terms are known, relatively few with 11, 12, 13 or more terms have been found. In some searches negative primes were accepted (see, for example, pp. 300–301 in [3]), but we shall restrict our attention to progressions consisting of positive primes only. The letter p, with or without a subscript, will always denote a prime. In particular, p_n will represent the nth prime so that $p_1 = 2, p_2 = 3$, etc. We shall denote the common difference of any given progression by d.

If $p \nmid d$, then any p consecutive terms of a progression constitute a complete residue system modulo p so that one of the terms is divisible by p. It is, therefore, easy to see that for a progression with exactly p_n terms either d is divisible by $P_n = 2 \cdot 3 \cdots p_n$ or $P_{n-1} | d$ *and* the first term of the progression is p_n. If a progression contains more than p_n terms, then $P_n | d$. For most of the known progressions with 10 or 11 terms $d = P_6 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$. Indeed, one might well expect to find arithmetic progressions of primes with up to 16 terms having common difference P_6 but, apparently, none is known with more than 12 terms. (The smallest is $23143 + 30030k$ ($k = 0, \ldots, 11$).)

A rather natural question suggests itself: If m is a given positive integer, how large must x_m be if one is to have a "reasonable" chance of finding a progression with m terms, none of which exceeds x_m? In Table 2 we give for each $m = 2, 3, \ldots, 17$ the arithmetic progression with m terms for which the last (mth) term q_m is the smallest known to date. For $m \leqslant 10$, q_m is *the* minimal value for the last term of a progression of length m. For $m = 11, 12, 13$ it is highly likely that the given value of q_m is indeed the smallest mth term that exists. For $m = 14, 15, 16, 17$ this is much less

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certain. In any event, these values of q_m indicate rather clearly that we may expect x_m to increase quite rapidly with m. One possible approach to evaluating x_m is to derive a formula (or approximation), say $F_m(x)$, for $N_m(x)$, the number of arithmetic progressions with m terms and largest member not exceeding x, and then determine y_m so that $F_m(x) \geqslant 1$ for $x \geqslant y_m$. If for any given m one can show that y_m exists (and, of course, $F_m(x)$ is a "good" approximation to $N_m(x)$), one will have verified the existence of arithmetic progressions of arbitrary length and consisting only of primes.

Several reasonably independent methods lead to explicit asymptotic formulas for $N_m(x)$. In what follows we shall discuss two of these formulas which we denote by $\overline{N}_m(x)$ and $N_m^*(x)$, respectively. The smallest positive integer y_m such that $\overline{N}_m(x) \geqslant 1$ for $x \geqslant y_m$ will furnish us with a (hopefully) reasonable approximation to the desired value x_m (see Table 2). Moreover, these formulas show that for all m, $\lim_{x\to\infty} N_m(x) = \infty$ and, hence, appear to confirm the stated conjecture. For $m \leqslant 3$, the formulas are known to be correct. For $m > 3$, however, all known "proofs" make use at some point of an unproved assumption. Thus, the use of these asymptotic formulas must, at present, be considered as being only heuristically justified.

The purpose of the present paper is to compare results obtained by these asymptotic formulas with actual computer counts of $N_m(x)$. For practical reasons the largest value of x considered was 50,000. The results indicate that: (a) for large values of m the limit 50,000 is far too low to permit any accurate prediction of the growth rate of $N_m(x)$; (b) the heuristic formulas are probably correct.

2. Some Heuristic Results. In 1922 Hardy and Littlewood published Part III of their famous series of "Partitio Numerorum" [6]. There, among other results, they prove a number of theorems based on plausible, but unproved, conjectures. Specifically, following Hardy and Littlewood, let $a_1, a_2, \ldots, a_m$ be given, distinct positive integers; and let

$$f_m(x) = \sum_{p=2}^{\infty} \Lambda(p)\Lambda(p + a_1) \cdots \Lambda(p + a_m)x^p,$$

where $\Lambda(n)$ is von Mangoldt's function ($\Lambda(n) = \log p$ if $n = p^k$, $\Lambda(n) = 0$ otherwise). Using heuristic reasoning, one is led to *Hypothesis* X (see p. 56 in [6]): If $m \geqslant 0$ and $r \longrightarrow 1$, then $f_m(r) \sim S_m/(1 - r)$. Here S_m is a constant, dependent on the a_j's, whose exact (and rather complicated) definition will not be needed in what follows.

Using Hypothesis X, Hardy and Littlewood prove six theorems. The first is the justly famous "m-tuples conjecture" which they call Theorem X_1. We quote this lengthy theorem in full, with some minor notational changes, in order to make the present paper self-contained.

THEOREM X_1. *Let $b_1, b_2, \ldots, b_m$ be m distinct integers, and $P(x, b_1, \ldots, b_m)$ the number of groups $n + b_1, n + b_2, \ldots, n + b_m$ between* 1 *and x and consisting wholly of primes. Then*

$$P(x) \sim G(b_1, b_2, \ldots, b_m) \cdot Li_m(x) \quad \textit{when } x \longrightarrow \infty,$$

where

$$G(b_1, \ldots, b_m) = \prod_{p \geq 2} \left(\frac{p}{p-1}\right)^{m-1} \left(\frac{p-\nu}{p-1}\right),$$

$\nu = \nu(p, b_1, \ldots, b_m)$ *is the number of distinct residues modulo p that occur in the set* $\{b_1, b_2, \ldots, b_m\}$, *and*

$$Li_m(x) = \int_2^x \frac{d\mu}{(\log \mu)^m}.$$

Further,

$$G(b_1, b_2, \ldots, b_m) = D_m \cdot H(b_1, b_2, \ldots, b_m),$$

where

$$D_m = \prod_{p>m} \left\{\left(\frac{p}{p-1}\right)^{m-1} \left(\frac{p-m}{p-1}\right)\right\},$$

$$H(b_1, b_2, \ldots, b_m) = \prod_{p \leq m} \left\{\left(\frac{p}{p-1}\right)^{m-1} \left(\frac{p-\nu}{p-1}\right)\right\} \prod_{\substack{p \mid \Delta \\ p>m}} \left(\frac{p-\nu}{p-m}\right),$$

and Δ *is the product of the differences of the* b_j*'s.*

On the basis of Theorem X_1 one of the present authors has proved (see [5]) the following result.

THEOREM. *Let* $m \geq 2$, *and denote by* $N_m(x)$ *the number of arithmetic progressions of m terms consisting only of primes, none larger than x. Define the constant*

$$C_m = \prod_{p \leq m} \left\{\frac{1}{p}\left(\frac{p}{p-1}\right)^{m-1}\right\} \prod_{p>m} \left\{\left(\frac{p}{p-1}\right)^{m-1} \left(\frac{p-m+1}{p}\right)\right\}, \tag{1}$$

and set

$$\overline{N}_m(x) = \frac{C_m}{2(m-1)} \cdot \frac{x^2}{\log^m x}. \tag{2}$$

Then (assuming Theorem X_1*)*

$$N_m(x) \sim \overline{N}_m(x). \tag{3}$$

The factor $1/2(m-1) \cdot x^2/\log^m x$ in (2) is the asymptotic value of the sum

$$S_m(x) = \sum (\log n_1 \cdot \log n_2, \ldots, \log n_m)^{-1}. \tag{4}$$

Here the summation is extended over all m-tuples $n_1, n_2, \ldots, n_m$ of positive integers such that $n_1 < n_2 < \cdots < n_m$ are in arithmetic progression and $2 \leq n_1, n_m \leq x$. Indeed, making use of a highly nontrivial application of the Euler-Maclaurin formula it is shown in [5] that $S_m(x)$ can be represented by an asymptotic series as follows:

$$S_m(x) = \frac{x^2}{2(m-1)\log^m x} \left\{1 + \sum_{j=1}^{N} \frac{a_j(m)}{\log^j x} + O\left(\frac{1}{\log^{N+1} x}\right)\right\}, \tag{5}$$

where all the $a_j(m)$ are computable and N may be taken arbitrarily large. The series itself in (5) does not converge.

At least two other methods lead, but still only heuristically, to (3). One, due to D. Zagier [13], is based on the assumption of the independence of the distribution of residue classes modulo distinct primes (regardless of the length of the intervals considered). The other is based on Vinogradov's version of the "circle method" of Hardy, Littlewood and Ramanujan. For $m = 3$, (3) (and, in fact, somewhat more) has been obtained by this approach without the use of any unproved hypotheses (see [5]); but for $m \geqslant 4$ the technical difficulties could not be overcome. However, if one proceeds formally, ignoring such difficulties as large error terms, etc., one obtains $N_m(x)$ as the product of a "singular series" and the sum in (4).

In view of the fact that several methods lead to (3), but that none of them (thus far) provide a convincing *proof* if $m \geqslant 4$, it appears desirable at this time to investigate to what extent actual calculations are in agreement with (3).

3. Auxiliary Considerations. Before we describe the results obtained a few remarks are in order. From (2) and (5), for $m \geqslant 2$, (3) is equivalent to

$$N_m(x) \sim N_m^*(x) = C_m S_m(x), \tag{6}$$

where $S_m(x)$ is given by (4). However, (6), is meaningful also for $m = 1$ which is not the case for (3). In fact, (6) is *true* both for $m = 1$ and $m = 2$ as we easily verify as follows.

If $m = 1$ then, from (1), $C_1 = 1$ and $N_1^*(x) = \Sigma_{n=2}^x (1/\log n) = \operatorname{li} x + O(1)$, where $\operatorname{li} x$ is the integral logarithm. Since every prime is an arithmetic progression of length one, $N_1(x) = \pi(x)$; and since, by the Prime Number Theorem, $\pi(x) \sim \operatorname{li} x$, it follows that $N_1(x) \sim N_1^*(x)$.

More can be said. For, by the Prime Number Theorem, $\pi(x) = \operatorname{li} x + O(xe^{-c\sqrt{\log x}})$ $= \operatorname{li} x + O(x \log^{-M} x)$, where M is any positive integer. Therefore, from the classical asymptotic series for $\operatorname{li} x$, we have

$$N_1(x) = \pi(x) = \frac{x}{\log x}\left\{1 + \frac{1!}{\log x} + \frac{2!}{\log^2 x} + \cdots + \frac{N!}{\log^N x} + O\left(\frac{1}{\log^{N+1} x}\right)\right\}.$$

If $m = 2$, then $C_2 = 1$ and

$$N_2^*(x) = \sum_{\substack{n_2=3\\ 2\leqslant n_1<n_2}}^{x} (\log n_1 \cdot \log n_2)^{-1} = \tfrac{1}{2}\left\{\left(\sum_{n=2}^{x} \frac{1}{\log n}\right)^2 - \sum_{n=2}^{x} \log^{-2} n\right\}$$

$$= \tfrac{1}{2}\{(\operatorname{li} x + O(1))^2 + O(\operatorname{li} x)\} = \tfrac{1}{2}\operatorname{li}^2 x + O(\operatorname{li} x)$$

$$= \tfrac{1}{2}\operatorname{li}^2 x(1 + O(\log x/x)),$$

where we have used the fact that $\operatorname{li} x \sim x/\log x$. On the other hand, since every pair of distinct primes is an arithmetic progression of length two, it is clear that $N_2(x)$ is the number of combinations of $\pi(x)$ primes taken two at a time. Therefore,

$$N_2(x) = \tfrac{1}{2}\pi(x)[\pi(x) - 1] = \tfrac{1}{2}\{\operatorname{li} x + O(xe^{-c\sqrt{\log x}})\}\{\operatorname{li} x + O(xe^{-c\sqrt{\log x}})\}$$

$$= \tfrac{1}{2}\operatorname{li}^2 x(1 + o(1));$$

and it follows that $N_2(x) \sim N_2^*(x)$. Again using the asymptotic series for $\operatorname{li} x$, we obtain

$$N_2(x) = \tfrac{1}{2}\pi(x)[\pi(x) - 1] = \tfrac{1}{2}[\operatorname{li} x + O(x \log^{-M-1} x)][\operatorname{li} x + O(x \log^{-M-1} x)]$$

$$= \tfrac{1}{2} \operatorname{li}^2 x\left[1 + O\left(\frac{1}{\log^M x}\right)\right]$$

$$= \frac{x^2}{2\log^2 x}\left\{1 + \frac{2}{\log x} + \frac{5}{\log^2 x} + \cdots + \frac{a_N(2)}{\log^N x} + O\left(\frac{1}{\log^{N+1} x}\right)\right\}.$$

We now call attention to the following rather unpleasant computational fact. For a relatively small value of x, say 10^4, $\log x \approx 9.2$, $2/\log x > .21$ and $5/\log^2 x > .05$. Hence, for $m = 2$ and $x = 10^4$, (5) is affected by an error in excess of 26% if we neglect the "corrective terms" in the asymptotic formula for $S_m(x)$. The situation is much worse for larger values of m.

If $m = 3$, (1) yields $C_3 = 2\prod_{p\neq 2}(1 - (p-1)^{-2}) = 2C_0$, where $C_0 = .66016\ldots$ the "twin primes" constant. From (2) and (3), $N_3(x) \sim (C_0/2)x^2/\log^3 x$. In fact, it is known (see [5]) that

$$N_3(x) = \frac{C_0}{2}\cdot\frac{x^2}{\log^3 x}\left\{1 + \frac{a_1}{\log x} + \frac{a_2}{\log^2 x} + O\left(\frac{1}{\log^3 x}\right)\right\},$$

where $a_1 = 3.5 - \log 2 > 2.8$ and $a_2 = 13 - 5\log 2 - \log^2 2 - \pi^2/12 \approx 8.231344049\ldots$. If one so desires, any number of coefficients of the asymptotic series represented by the bracket in the expression for $N_3(x)$ just given can be computed. If $x = 10^4$, then $a_1/\log x > .3$. If we neglect this term, our error exceeds 30%. As in the case $m = 2$ this example shows that if we want to compare in some meaningful way actual values of $N_m(x)$ with values obtained by our formulas, we have to choose among the following options:

(i) formulas (2) and (3) may be used *if* large enough values of x are utilized so that the neglected terms of the asymptotic series are indeed negligible;

(ii) formula (6) may be used;

(iii) the first terms of the asymptotic expansion in (5) may be used so that (6) becomes

$$(7)\quad N_m(x) \sim \frac{C_m}{2(m-1)}\cdot\frac{x}{\log^m x}\left\{1 + \frac{a_1(m)}{\log x} + \cdots + \frac{a_N(m)}{\log^N x} + O\left(\frac{1}{\log^{N+1} x}\right)\right\}.$$

The need for these corrective terms, at moderate values of x, was already recognized by Hardy and Littlewood (see pp. 37–38 in [6]).

The values of x required by (i) are beyond the practical range of present day com puters for counting $N_m(x)$. On the other hand, the computation of the coefficients $a_j(m)$ required by (iii), quite easy if m is small, becomes very difficult for large values of m. These considerations led us to rely primarily on (ii) in the present paper.

4. A Description of Our Tables and Some Comments. Using the CDC 6400 at the Temple University Computer Center, the values of $N_m(x)$, $\overline{N}_m(x)$ (see (2)), and $N_m^*(x)$ (see (4) and (6)) were calculated for $m = 3(1)10$, $x = 1000(1000)10000$ and

$x = 10000(10000)50000$ and for $m = 3, 4, 5$ are given in Table. 1. (An extension of Table 1 which covers the cases $m = 6$ to $m = 10$ is available from the authors upon request.) The required values of C_m were computed by truncating the infinite product in (1) at $p = 999983$, and estimating the error thus incurred. The results (correct to six significant digits for $m < 15$ and five significant digits for $15 \leqslant m \leqslant 20$) appear in Table 3. The values of $S_m(x)$ were obtained by a rather straightforward application of definition (4). Thirteen significant digits were carried in the calculations, and the results were rounded to seven digits to minimize roundoff error.

The ratios $N_m(x)/N_m^*(x)$, $N_m(x)/\overline{N}_m(x)$ and $N_m^*(x)/\overline{N}_m(x)$ are also given in Table 1. If the conjectural (for $m \geqslant 4$) formulas are correct these ratios will *all* approach unity as $x \longrightarrow \infty$. Otherwise, only $N_m^*(x)/\overline{N}_m(x)$ may be expected to converge to one.

Since (3) and (6) are known to be true for $m = 3$ but (as of this writing) are still only conjectural for $m \geqslant 4$, the speed of convergence of $N_m(x)/N_m^*(x)$ and $N_m(x)/\overline{N}_m(x)$ to unity in the former case may be indicative of what could be expected in the cases $m \geqslant 4$ if (3) and (6) are indeed true for *all m*. In this context we observe that the ratio $N_3(x)/N_3^*(x)$ (the more significant of the two) increases from .841 for $x = 1000$ to .956 ... for $x = 10000$ to .980 ... for $x = 50000$. For $m = 4$ the corresponding values increase from .693 ... to .927 ... to .970 ... . However, already for $m = 6$ the figures deteriorate to .371 ... , .695 ..., .895 ..., respectively. For $m = 7$ the figures are .079 ..., .439 ..., .789 ...; for $m = 8$ they are 0, .270 ..., .680 ...; for $m = 9$ they are 0, .221 ..., .464 ... . An eventual convergence to unity is still suggested, but the rate of convergence appears to diminish rapidly as m increases.

TABLE 1

m = 3

x	$N_3(x)$	$N_3^*(x)$	$\overline{N}_3(x)$	N_3/N_3^*	$N_3/\overline{N}_3$	$N_3^*/\overline{N}_3$
1000	1500	1782.4	1001.4	.84155	1.4979	1.7799
2000	4457	4980.5	3006.7	.89489	1.4824	1.6565
3000	8478	9255.5	5788.4	.91600	1.4647	1.5990
4000	13356	14472	9256.4	.92290	1.4429	1.5634
5000	19174	20548	13356	.93315	1.4356	1.5385
6000	25679	27426	18048	.93630	1.4228	1.5196
7000	33319	35064	23305	.95024	1.4297	1.5046
8000	41029	43427	29102	.94478	1.4098	1.4922
9000	49721	52488	35422	.94728	1.4037	1.4818
10000	59504	62224	42247	.95629	1.4085	1.4729
20000	186647	193195	135930	.96611	1.3731	1.4213
30000	368304	378543	271156	.97295	1.3583	1.3690
40000	599294	612470	443850	.97849	1.3502	1.3799
50000	873953	891421	651487	.98040	1.3415	1.3683

m = 4

x	$N_4(x)$	$N_4^*(x)$	$\bar{N}_4(x)$	N_4/N_4^*	$N_4/\bar{N}_4$	$N_4^*/\bar{N}_4$
1000	318	458.36	209.22	.69378	1.5199	2.1908
2000	914	1126.1	570.88	.81162	1.6010	1.9726
3000	1670	1956.4	1043.4	.85361	1.6005	1.8750
4000	2555	2925.1	1610.7	.87347	1.5863	1.8161
5000	3568	4017.8	2263.1	.88805	1.5766	1.7754
6000	4704	5224.3	2994.1	.90040	1.5711	1.7449
7000	6004	6537.3	3798.9	.91843	1.5805	1.7208
8000	7304	7950.6	4673.4	.91867	1.5629	1.7013
9000	8682	9459.6	5614.6	.91780	1.5463	1.6848
10000	10257	11060	6619.8	.92738	1.5494	1.6708
20000	29811	31516	19809	.94589	1.5049	1.5910
30000	56528	58943	37961	.95902	1.4891	1.5528
40000	89345	92400	60450	.96693	1.4780	1.5285
50000	127397	131323	86899	.97010	1.4660	1.5112

m = 5

x	$N_5(x)$	$N_5^*(x)$	$\bar{N}_5(x)$	N_5/N_5^*	$N_5/\bar{N}_5$	$N_5^*/\bar{N}_5$
1000	58	93.881	32.991	.61780	1.7580	2.8456
2000	141	199.61	81.812	.70639	1.7235	2.4398
3000	245	321.74	141.95	.76148	1.7259	2.2665
4000	352	457.95	211.53	.76864	1.6641	2.1650
5000	464	606.69	289.43	.76480	1.6032	2.0962
6000	604	766.88	374.89	.78761	1.6111	2.0456
7000	780	937.67	467.37	.83185	1.6689	2.0063
8000	923	1118.4	566.42	.82529	1.6295	1.9745
9000	1091	1308.5	671.69	.83376	1.6243	1.9481
10000	1283	1507.6	782.89	.85100	1.6388	1.9257
20000	3484	3925.5	2178.7	.88752	1.5991	1.8018
30000	6383	6995.4	4011.0	.91246	1.5914	1.7441
40000	9894	10614	6213.9	.93216	1.5922	1.7081
50000	13835	14721	8748.4	.93982	1.5814	1.6827

For "large" values of m the upper bound $x = 50000$ of our computations is totally inadequate for drawing meaningful quantitative conclusions. This is clearly illustrated by our findings for the case $m = 10$. $N_{10}(x)/N_{10}^*(x)$ varies from 0 for $x = 1000$ to .143 ... for $x = 10000$ and then *decreases* to .082 ... for $x = 50000$. This erratic behavior is very simply explained. The first progression of 10 terms is $199 + 210k$ ($k = 0, \ldots, 9$); the next two are $34913 + 2100k$ and $52879 + 420k$ ($k = 0, \ldots, 9$) with last terms 53813 and 56659, respectively. Consequently, the ratio $N_{10}(x)/N_{10}^*(x)$ almost triples from about .08 for $x = 50000$ to about .23 for $x = 57000$.

Table 2 has already been described in Section 1. For $5 \leqslant m \leqslant 20$, y_m correct to four significant digits, is the smallest integer such that $\overline{N}_m(x) > 1$ for $x \geqslant y_m$. The value of y_m is not tabulated for $m = 2, 3, 4$ since $\overline{N}_m(x) > 1$ for these values of m if $x \geqslant 2$.

TABLE 2

m	Progression with minimal last term	q_m	y_m
2	2,3	3	-
3	3,5,7	7	-
4	5,11,17,23	23	-
5	5,11,17,23,29	29	29
6	7 + 30k *	157	92
7	7 + 150k	907	497
8	199 + 210k	1669	1406
9	199 + 210k	1879	5086
10	199 + 210k	2089	24310
11	110437 + 13860k	249037	177300
12	110437 + 13860k	262897	829800
13	4943 + 60060k	725663	5582000
14	46883579 + 2462460k	78895559	2.332×10^7
15	53297929 + 9699690k	189093589	1.137×10^8
16	53297929 + 9699690k	198793279	6.793×10^8
17	3430751869 + 87297210k	4827507229	5.774×10^9
18	-	-	3.303×10^{10}
19	-	-	2.564×10^{11}
20	-	-	1.261×10^{12}

* In each case k = 0,1,...,m-1.

The fact that $y_5 = q_5$ is, of course, coincidental. In fact, one cannot expect y_m to approximate q_m too closely. Indeed, this is already precluded by the fact that $\overline{N}_m(x)$ has been used to find y_m rather than $N_m^*(x)$ *and* all the corrective terms of the asymptotic series (5) have been neglected. Furthermore, for small values of x the arithmetic progressions of length m with largest term not exceeding x are distributed in a very irregular manner as we saw in the case $m = 10$. Under these circumstances it is rather remarkable how well y_m seems to indicate the correct order of magnitude of q_m. Table 2 also "explains" why it is necessary to go to progressions with such large terms in order to find arithmetic progressions with a modest length like $m = 16$ or 17. Finally, Table 2 indicates that if one is seeking an arithmetic progression of, say, 20 terms one must expect its last term to have about 13 digits.

TABLE 3

m	C_m
3	1.32032
4	2.85825
5	4.15118
6	10.1318
7	17.2986
8	53.9720
9	148.552
10	336.034
11	511.422
12	1312.32
13	2364.60
14	7820.61
15	22939
16	55651
17	91555
18	256480
19	510990
20	1901000

5. Final Conclusions. While the upper bound, $x = 50000$, of our computations is far too small to permit any quantitative inferences, especially for $m > 6$, the validity of the heuristic formulas (3) and (6) is strongly suggested by our computer data. This conclusion is reinforced when one compares the rates of convergence of the ratios $N_m(x)/N_m^*(x)$ and $N_m(x)/\overline{N}_m(x)$ to one in the case $m = 3$ (when (3) and (6) are known to be valid) with those in the cases $m = 4$ and $m = 5$ *and* when one observes the (overall) agreement of the orders of magnitude of q_m and y_m.

Department of Mathematics
Temple University
Philadelphia, Pennsylvania 19122

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